Some Time Domain Electromagnetics

We will concentrate on the fields associated with moving charges in free space. Starting from Maxwell's equations in free space:

$$\begin{cases} \bar{\nabla}\times\bar{E} = -\dfrac{\partial \bar{B}}{\partial t} & \bar{\nabla}\cdot\bar{B} = 0 \\ \bar{\nabla}\times\bar{H} = \dfrac{\partial \bar{D}}{\partial t} + \bar{J} & \bar{\nabla}\cdot\bar{D} = \rho \end{cases}$$

we already know that the electromagnetic field can be represented in terms of a magnetic vector potential and an electric scalar potential, $\bar{A}, \phi$:

$$\bar{B} = \bar{\nabla}\times\bar{A}$$

$$\bar{E} = -\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t}$$

where $\bar{A}, \phi$ satisfy wave equations:

$$\bar{\nabla}^2\bar{A} - \mu_0\varepsilon_0 \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu\bar{J}$$

$$\nabla^2\phi - \mu_0\varepsilon_0 \frac{\partial^2 \phi}{\partial t^2} = -\rho/\varepsilon_0$$

$\therefore$ each rectangular component of $\bar{A}$ and ϕ satisfies a scalar wave equation:

$$\boxed{\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi q(\bar{r},t)}$$

(Morse / Feshbach, sec. 7.3)

Green's Function for the Scalar Wave Equation

let ψ satisfy the inhomogeneous scalar wave equation:

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -4\pi q(\bar{r}, t) \qquad (1)$$

where $q(r,t)$ is the source density function

as well as the PDE, (1), it is necessary to state boundary and initial conditions in order to obtain a unique solution:

$$\text{let} \quad \left. \begin{aligned} \psi(t_0) &= \psi_0(\bar{r}) \\ \left.\frac{\partial \psi}{\partial t}\right|_{t=t_0} &= v_0(\bar{r}) \end{aligned} \right\} \qquad (2)$$

be the initial conditions (Cauchy)

The Green's Function for this problem is the function satisfying the simpler differential equation:

$$\nabla^2 G - \frac{1}{c^2} \frac{\partial^2 G}{\partial t^2} = -4\pi\, \delta(\bar{r} - \bar{r}_0)\, \delta(t - t_0) \qquad (3)$$

i.e. where the source density is an impulse at time $t = t_0$ located at $\bar{r} = \bar{r}_0$

Note: G satisfies the homogeneous form of the boundary conditions satisfied by ψ on the boundary of the region of interest.

we assume that $G = 0$, $\frac{\partial G}{\partial t} = 0$ for $t < t_0$

this is a statement of causality: i.e. if an impulse occurs at $t = t_0$, no effects of the impulse should be present at an earlier time.

It can be proved that a <u>Reciprocity relation</u> exists for the Green's Function:

$$\boxed{G(\bar{r}, t \mid \bar{r}_0, t_0) = G(\bar{r}_0, -t_0 \mid \bar{r}, -t)} \qquad (4)$$

we see that for $t > t_0$ the effect at $\bar{r}, t$ due to an impulse at $\bar{r}_0, t_0$ is the same as the effect at $\bar{r}_0, -t_0$ due to an impulse at $\bar{r}, -t$.

We will need this reciprocity relationship in order to find an expression to our inhomogeneous wave equation (1) with initial conditions specified by (2)

Consider (1) in the $(\bar{r}_0, t_0)$ coordinates:

$$\nabla_0^2 \psi(\bar{r}_0, t_0) - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t_0^2} = -4\pi q(\bar{r}_0, t_0) \qquad (5)$$

and consider $G(\bar{r}_0, t_0' | \bar{r}, t')$: $t' < t_0'$

$$\nabla_0^2 G(\bar{r}_0, t_0' | \bar{r}, t') - \frac{1}{c^2} \frac{\partial^2}{\partial t_0'^2} G(\bar{r}_0, t_0' | \bar{r}, t') = -4\pi \, \delta(\bar{r}_0 - \bar{r}) \, \delta(t_0' - t')$$

by reciprocity $G(\bar{r}_0, t_0' | \bar{r}, t') = G(\bar{r}, -t' | \bar{r}_0, -t_0)$

now let $t_0 = -t_0'$ $\Rightarrow$ $\frac{\partial^2}{\partial t_0'^2} = \frac{\partial^2}{\partial t_0^2}$

$$\delta(t_0' - t') = \delta(-t_0 - t')$$

also let $t = -t'$ $\Rightarrow$ $\delta(-t_0 - t') = \delta(t - t_0)$

$\therefore$ we have: $\left(\delta(\bar{r}_0 - \bar{r}) = \delta(\bar{r} - \bar{r}_0) \right)$

$$\begin{cases} \nabla_0^2 G(\bar{r}, t | \bar{r}_0, t_0) - \frac{1}{c^2} \frac{\partial^2}{\partial t_0^2} G(\bar{r}, t | \bar{r}_0, t_0) \\ \qquad = -4\pi \, \delta(\bar{r} - \bar{r}_0) \, \delta(t - t_0) \end{cases} \quad (6)$$

with: $-t < -t_0$ $\Rightarrow$ $t > t_0$

now take $\int_0^{t^+} \int_{V_0} G(\bar{r}, t | \bar{r}_0, t_0) \times (5) - \psi(\bar{r}_0, t_0) \times (6) \, dV_0 \, dt_0$

where $t^+ \Rightarrow t + \epsilon$ and ϵ is arbitrarily small

we get:

$$\int_0^{t^+} \int_{V_0} G \nabla_0^2 \psi - \psi \nabla_0^2 G + \frac{1}{c^2} \left(\frac{\partial^2 G}{\partial t_0^2} \psi - G \frac{\partial^2 \psi}{\partial t_0^2} \right) dV_0 \, dt_0 \quad (7)$$

$$= -4\pi \int_0^{t^+} \int_{V_0} q(\bar{r}_0, t_0) \, G(\bar{r}, t | \bar{r}_0, t_0) \, dV_0 \, dt_0 + 4\pi \, \psi(\bar{r}, t)$$

now Green's theorem states: (prove this!)

$$\oiint_S (u\nabla v - v\nabla u)\cdot d\bar{S} = \iiint_V u\nabla^2 v - v\nabla^2 u \; dV$$

and we have the identity:

$$\frac{\partial}{\partial t_0}\left[\frac{\partial G}{\partial t_0}\psi - G\frac{\partial \psi}{\partial t_0}\right] = \frac{\partial^2 G}{\partial t_0^2}\psi - G\frac{\partial^2 \psi}{\partial t_0^2}$$

$\therefore$ we get from ⑦

$$\left.\begin{aligned} 4\pi\psi(\bar{r},t) = {} & 4\pi\int_0^{t^+}\int_{V_0} q(\bar{r}_0,t_0)\,G(\bar{r},t\,|\,\bar{r}_0,t_0)\,dV_0\,dt_0 \\ & + \int_0^{t^+}\oint_{S_0}(G\nabla_0\psi - \psi\nabla_0 G)\cdot d\bar{S}_0\,dt_0 \\ & + \frac{1}{c^2}\int_{V_0}\left[\frac{\partial G}{\partial t_0}\psi - G\frac{\partial\psi}{\partial t_0}\right]_0^{t^+} dV_0 \end{aligned}\right\} \quad ⑧$$

now recall we started with $G(\bar{r}_0,t_0'\,|\,\bar{r},t')$
and the initial conditions were the homogeneous ones

$$G(\bar{r}_0,t_0'\,|\,\bar{r},t') = \frac{\partial}{\partial t_0'}G = 0 \quad \text{for } t_0' < t'$$
$$\Rightarrow \quad t_0 > t$$

in the third integral in ⑧ we have $t_0 = t^+ = t+\epsilon$

$$\therefore \left[\frac{\partial G}{\partial t_0}\psi - G\frac{\partial\psi}{\partial t_0}\right]\Bigg|_{t_0=t^+} = 0$$

and $$\left[\frac{\partial G}{\partial t_0}\psi - G\frac{\partial \psi}{\partial t_0}\right]_{t_0=0} = \left(\frac{\partial G}{\partial t_0}\right)_{t_0=0}\psi_0(\bar{r}_0) - G_{t_0=0}\, v_0(\bar{r})$$

Thus we get

$$\left.\begin{aligned} 4\pi\,\psi(\bar{r},t) &= 4\pi\int_0^{t^+}\int_{V_0} G(\bar{r},t\,|\,\bar{r}_0,t_0)\, q(\bar{r}_0,t_0)\, dV_0\, dt_0 \\ &\quad + \int_0^{t^+}\oint_{S_0}\left[G\,\nabla_0\psi - \psi\,\nabla_0 G\right]\cdot d\bar{S}_0\, dt_0 \\ &\quad - \frac{1}{c^2}\int_{V_0}\left[\left(\frac{\partial G}{\partial t_0}\right)_{t_0=0}\psi_0(\bar{r}_0) - G_{t_0=0}\, v_0(\bar{r}_0)\right] dV_0 \end{aligned}\right\} \quad (9)$$

First integral represents effect of source term $q(\bar{r}_0,t_0)$

second integral represents effect of boundary conditions

last integral represents the contribution to the field due to any initial displacement and initial velocity

Notice that we can simulate the initial condition term by specifying an impulsive source term of the form:

$$q(\bar{r}_0,t_0) = \frac{1}{c^2}\left[\psi_0(\bar{r}_0)\,\delta'(t_0) + v_0(\bar{r}_0)\,\delta(t_0)\right] \quad (10)$$

where:
$$\int_a^b f(x)\,\delta'(x)\,dx = \begin{cases} -f'(0) & ; \text{ if } x=0 \in (a,b) \\ 0 & ; \text{ if } x=0 \notin (a,b) \end{cases}$$

thus in (10) we see that an initial velocity can be simulated by a $v_0(\bar{r}_0)\,\delta(t_0)$ source

an initial displacement by $\psi_0(\bar{r}_0)\,\delta'(t_0)$

$$\psi_0(\bar{r}_0)\,\delta'(t_0) = \lim_{\epsilon \to 0} \left\{ \psi_0(\bar{r}_0) \left[\frac{\delta(t_0+\epsilon) - \delta(t_0-\epsilon)}{2\epsilon} \right] \right\}$$

i.e. first a positive impulse and then a negative one to reduce the velocity to zero.

Now in order to use (9) we must solve for an appropriate Green's Function.

We now consider the form of this Green's Function for the scalar wave equation in three dimensions for the case of the infinite domain.

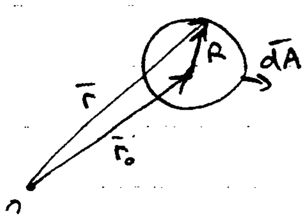
3-D infinite domain scalar wave equations Green's function

let the infinite domain G.F. be called $g(\bar{r},t|\bar{r}_0,t_0)$ and satisfy the scalar wave equation:

$$\nabla^2 g - \frac{1}{c^2}\left(\frac{\partial^2 g}{\partial t^2}\right) = -4\pi\,\delta(\bar{r}-\bar{r}_0)\,\delta(t-t_0) \qquad (11)$$

define $R = |\bar{r}-\bar{r}_0|$ and integrate both sides of (11) over a small spherical volume surrounding $\bar{r}=\bar{r}_0$ (i.e. $R=0$):

$$\iiint_V \nabla^2 g\, dv - \frac{1}{c^2}\iiint_V \frac{\partial^2 g}{\partial t^2}\, dv = -4\pi\,\delta(t-t_0)$$



by symmetry g depends only on R

$\therefore$ First term becomes:

$$\iiint_{R\to\epsilon} \nabla^2 g(R)\, dv = \oint\!\!\oint_{R\to\epsilon} \nabla g(R)\cdot d\bar{A} = \left.\frac{dg}{dR}\right|_{R=\epsilon} 4\pi\epsilon^2$$

$$\Rightarrow -4\pi\,;\ \epsilon\to 0$$

(aposteriori we can neglect the second term it is a lesser singularity than the first term)

$$\therefore\ \frac{dg}{dR} \to -\frac{1}{R^2} \qquad R\to 0$$

$$\Longrightarrow\ g(R) = \frac{1}{R}$$

$$\boxed{g(R,t) = \frac{\delta(t-t_0)}{R}} \quad (12) \qquad R\to 0$$

we can solve the equation for $R, t-t_0 \neq 0$ using the debye Potential:

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial g}{\partial R}\right) - \frac{1}{c^2}\frac{\partial^2 g}{\partial t^2} = 0$$

let $g = \frac{u}{R}$:

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial(u/R)}{\partial R}\right) - \frac{1}{c^2 R}\frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left[R^2\left(\frac{1}{R}\frac{\partial u}{\partial R} - \frac{u}{R^2}\right)\right] - \frac{1}{c^2 R}\frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left[R\frac{\partial u}{\partial R} - u\right] - \frac{1}{c^2 R}\frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{1}{R^2}\left[\frac{\partial u}{\partial R} + R\frac{\partial^2 u}{\partial R^2} - \frac{\partial u}{\partial R}\right] - \frac{1}{c^2 R}\frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{\partial^2 u}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

by method of characteristics we have:

$$u(R,t) = F(t - R/c) + h(t + R/c)$$

where F, h are arbitrary Functions.

$$\therefore \quad g = \frac{F(t - R/c)}{R} \qquad h \equiv 0 \text{ by causality}$$

but as $R \to 0$, (12) says that $g(R,t) \to \frac{\delta(t - t_0)}{R}$

$$\therefore \quad \boxed{g(R,t\,|\,t_o) = \frac{\delta((t-t_o) - R/c)}{R}} \qquad (13)$$

Free space Green's Function

Now consider the solution to the problem:

$$\nabla^2 \psi(\bar{r},t) - \frac{1}{c^2}\frac{\partial^2 \psi(\bar{r},t)}{\partial t^2} = -q(\bar{r},t)$$

infinite domain
$-\infty \le x \le \infty$
$-\infty \le y \le \infty$
$-\infty \le z \le \infty$

$$\left.\begin{aligned} &\psi(0) = 0 \\ &\left.\frac{\partial \psi}{\partial t}\right|_{t=0} = 0 \end{aligned}\right\} \text{ no initial conditions}$$

then using (9),(13) we have:

$$\psi(\bar{r},t) = \frac{1}{4\pi}\int_0^{t^+}\int_{V_o} \frac{\delta((t-t_o)-R/c)}{R}\, q(\bar{r}_o,t_o)\, dV_o\, dt_o \qquad t^+ = t+\epsilon$$

$$\boxed{\psi(\bar{r},t) = \frac{1}{4\pi}\int_{V_o} \frac{q(\bar{r}_o\,,\,t - R/c)}{R}\, dV_o} \qquad (14)$$

retarded potential solution.

Proof of the time domain Reciprocity Relation

note: $G(\bar r, t \mid \bar r_0, t_0) \neq G(\bar r_0, t_0 \mid \bar r, t)$
since if $t > t_0$ $G(\bar r_0, t_0 \mid \bar r, t) = 0$

the right reciprocity relation is

$$\boxed{G(\bar r, t \mid \bar r_0, t_0) = G(\bar r_0, -t_0 \mid \bar r, -t)}$$

In order to visualize this consider the case where $t_0 = 0$, then we have

$$G(\bar r, t \mid \bar r_0, 0) = G(\bar r_0, 0 \mid \bar r, -t)$$

i.e. field at $(\bar r, t)$ due to disturbance at $\bar r_0$ at time zero $(t>0)$ is the same as the field at $(\bar r_0, 0)$ due to disturbance at $\bar r$, $-t$ before.

Proof:

$$\nabla^2 G(\bar r, t \mid \bar r_0, t_0) - \frac{1}{c^2}\left[\frac{\partial^2 G(\bar r, t \mid \bar r_0, t_0)}{\partial t^2}\right] = -4\pi\,\delta(\bar r - \bar r_0)\,\delta(t - t_0)$$

$$\nabla^2 G(\bar r, -t \mid r_1, -t_1) - \frac{1}{c^2}\left[\frac{\partial^2 G(\bar r, -t \mid \bar r_1, -t_1)}{\partial t^2}\right] = -4\pi\delta(\bar r - \bar r_1)\,\delta(t - t_1)$$

multiplying first by $G(\bar r, -t \mid \bar r_1, -t_1)$ second by $G(\bar r, t \mid \bar r_0, t_0)$, subtracting and integrating over region V and $-\infty < t < t'$ $t' > t_0$, $t' > t_1$ we get:

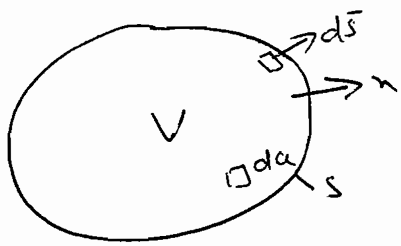
$$\int_{-\infty}^{t'} dt \int_V dv \Big\{ G(\bar{r},t|\bar{r}_0,t_0)\,\nabla^2 G(\bar{r},-t|\bar{r}_1,-t) - G(\bar{r},-t|\bar{r}_1,-t_1)\,\nabla^2 G(\bar{r},t|\bar{r}_0,t_0)$$

$$+ \frac{1}{c^2} G(\bar{r},t|\bar{r}_0,t_0)\frac{\partial^2}{\partial t^2} G(\bar{r},-t|\bar{r}_1,-t_1) - \frac{1}{c^2} G(\bar{r},-t|\bar{r}_1,-t_1)\frac{\partial^2}{\partial t^2} G(\bar{r},t|\bar{r}_0,t$$

$$= 4\pi \Big\{ G(\bar{r}_0,-t_0|\bar{r}_1,-t_1) - G(\bar{r}_1,t_1|\bar{r}_0,t_0) \Big\}$$

Now recall <u>Green's second identity</u>:

$$\int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, dv = \oint_S \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) da$$



$$= \oint_S (\psi \nabla \phi - \phi \nabla \psi) \cdot d\bar{s}$$

and if we use the identity:

$$\frac{\partial}{\partial t}\left[G(\bar{r},t|\bar{r}_0,t_0)\frac{\partial}{\partial t} G(\bar{r},-t|\bar{r}_1,-t_1) - G(\bar{r},-t|\bar{r}_1,-t_1)\frac{\partial}{\partial t} G(\bar{r},t|\bar{r}_0,t_0)\right]$$

$$\equiv G(\bar{r},t|\bar{r}_0,t_0)\frac{\partial^2}{\partial t^2} G(\bar{r},-t|\bar{r}_1,-t_1) - G(\bar{r},-t|\bar{r}_1,-t_1)\frac{\partial^2}{\partial t^2} G(\bar{r},t|\bar{r}_0,t_0)$$

then the above equation becomes:

$$\int_{-\infty}^{t'} dt \int_S d\bar{S} \cdot \left[G(\bar{r},t|\bar{r}_0,t_0)\nabla G(\bar{r}_1-t|\bar{r}_1,-t_1) - G(\bar{r},-t|\bar{r}_1,-t_1)\nabla G(\bar{r},t|\bar{r}_0,t_0) \right]$$

$$+ \frac{1}{c^2}\int_V dV \left[G(\bar{r},t|\bar{r}_0,t_0)\frac{\partial G(\bar{r},-t|\bar{r}_1,-t_1)}{\partial t} - G(\bar{r}_1-t|\bar{r}_1,-t_1)\frac{\partial G(\bar{r},t|\bar{r}_0,t_0)}{\partial t} \right]_{t=-\infty}^{t=t'}$$

$$= 4\pi \left\{ G(\bar{r}_0,-t_0|\bar{r}_1,-t_1) - G(\bar{r}_1,t_1|\bar{r}_0,t_0) \right\}$$

Since G satisfie homogeneous B.C.'s on S the first part of the L.H.S. $\to 0$.

For the second part on the L.H.S:

at $t=-\infty$: $\begin{cases} G(\bar{r},-\infty|\bar{r}_0,t_0) = 0 \\ \left.\dfrac{\partial}{\partial t} G(\bar{r},t|\bar{r}_0,t_0)\right|_{t\to-\infty} = 0 \end{cases}$

at $t=t'$ $(t'>t_0,\ t'>t_1)$: $\begin{cases} G(\bar{r},-t'|\bar{r}_1,-t_1) = 0 \\ \left.\dfrac{\partial}{\partial t} G(\bar{r},-t|\bar{r}_1,-t_1)\right|_{t=t'} = 0 \end{cases}$

Both of these are due to <u>causality</u> since $-\infty < t_0$

$-t' < -t_1$

Thus we have proved the appropriate causal reciprocity relation

$$G(\bar{r}_0, -t_0 \mid \bar{r}_1, -t_1) = G(\bar{r}_1, t_1 \mid \bar{r}_0, t_0)$$

or changing variables:

$$\boxed{G(\bar{r}_1, t \mid \bar{r}_0, t_0) = G(\bar{r}_0, -t_0 \mid \bar{r}_1, -t)}$$